

A Picard-S Iterative Scheme for Approximating Fixed Point of Weak-Contraction Mappings

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Abstract We study the convergence analysis of a Picard-S iteration method for a particular class of weak-contraction mappings. Furthermore, we prove a data dependence result for fixed point of the class of weak-contraction mappings with the help of the Picard-S iteration methods.

Key words Picard-S iterative scheme – Weak-Contraction mappings – Convergence – Rate of Convergence- Data Dependence

1 Introduction

Most of the problems that arise in different disciplines of science can be formulated by the equations in the form

$$Fx = 0, \quad (1)$$

where F is some function. The equations given by (1) can easily be reformulated as the fixed point equations of type

$$Tx = x. \quad (2)$$

where T is a self-map of an ambient space X and $x \in X$. These equations are often classified as linear or nonlinear, depending on whether the mappings used in the equation is linear with respect to the variables. Over the years, a considerable attention has been paid to solving such equations by using different techniques such as direct and iterative methods. In case of linear equations, both direct and iterative methods are used to obtain solutions of

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the equations. But in case of nonlinear equations, due to various reasons, direct methods can be impractical or fail in solving equations, and thus iterative methods become a viable alternative. Nonlinear problems are of importance interest to mathematicians, physicists and engineers and many other scientists, simply because most systems are intrinsically nonlinear in nature. That is why, researchers in various disciplines of sciences are often faced with the solving such problems. It would be hard to fudge that the role of iterative approximation of fixed points have played in the recent progress of nonlinear science. Indeed, the phrase iterative approximation has been introduced to describe repetition-based researches into nonlinear problems that are inaccessible to analytic methods. For this reason, the iterative approximation of fixed points has become one of the major and basic tools in the theory of equations, and as a result, numerous iterative methods have been introduced or improved and studied for many years in detail from various points of aspects by a wide audience of researchers, see, [1-14, 16-42, 44, 45].

In this paper, we show that a Picard-S iteration method [14] can be used to approximate fixed point of weak-contraction mappings. Also, we show that this iteration method is equivalent and converges faster than CR iteration method [9] for the aforementioned class of mappings. Furthermore, by providing an example, it is shown that the Picard-S iteration method converges faster than CR iteration method and hence also faster than all Picard [31], Mann [25], Ishikawa [17], Noor [26], SP [30], S [2] and some other iteration methods in the existing literature when applied to weak-contraction mappings. Finally, a data dependence result is proven for fixed point of weak-contraction mappings with the help of the Picard-S iteration method.

Throughout this paper the set of all positive integers and zero is shown by \mathbb{N} . Let B be a Banach space, D be a nonempty closed convex subset of B and T a self-map of D . An element x_* of D is called a fixed point of T if and only if $Tx_* = x_*$. The set of all fixed point of T denoted by F_T . Let $\{a_n^i\}_{n=0}^\infty$, $i \in \{0, 1, 2\}$ be real sequences in $[0, 1]$ satisfying certain control condition(s).

Renowned Picard iteration method [31] is formulated as follow

$$\begin{cases} p_0 \in D, \\ p_{n+1} = Tp_n, n \in \mathbb{N}, \end{cases} \quad (3)$$

and generally used to approximate fixed points of contraction mappings satisfying: for all $x, y \in B$ there exists a $\delta \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \delta \|x - y\|. \quad (4)$$

The following iteration methods are known as Noor [26] and SP [30] iterations, respectively:

$$\begin{cases} \omega_0 \in D, \\ \omega_{n+1} = (1 - a_n^0) \omega_n + a_n^0 T \varpi_n, \\ \varpi_n = (1 - a_n^1) \omega_n + a_n^1 T \rho_n, \\ \rho_n = (1 - a_n^2) \omega_n + a_n^2 T \omega_n, \end{cases} \quad n \in \mathbb{N}, \quad (5)$$

$$\begin{cases} q_0 \in D, \\ q_{n+1} = (1 - a_n^0) r_n + a_n^0 T r_n, \\ r_n = (1 - a_n^1) s_n + a_n^1 T s_n, \\ s_n = (1 - a_n^2) q_n + a_n^2 T q_n, \end{cases} \quad n \in \mathbb{N}, \quad (6)$$

Remark 1 (i) If $a_n^2 = 0$ for each $n \in \mathbb{N}$, then the Noor iteration method reduces to iterative method of Ishikawa [17].

(ii) If $a_n^2 = 0$ for each $n \in \mathbb{N}$, then the SP iteration method reduces to iterative method of Thianwan [42].

(iii) When $a_n^1 = a_n^2 = 0$ for each $n \in \mathbb{N}$, then both Noor and SP iteration methods reduce to an iteration method due to Mann [25].

Recently, Gürsoy and Karakaya [14] introduced a Picard-S iterative scheme as follows:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = T y_n, \\ y_n = (1 - a_n^1) T x_n + a_n^1 T z_n, \\ z_n = (1 - a_n^2) x_n + a_n^2 T x_n, \end{cases} \quad n \in \mathbb{N}, \quad (7)$$

The following definitions and lemmas will be needed in obtaining the main results of this article.

Definition 1 [5] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits a and b , respectively. Assume that there exists

$$\lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|} = l. \quad (8)$$

(i) If $l = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b .

(ii) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 2 [5] Assume that for two fixed point iteration processes $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p , the following error predictions

$$\|u_n - p\| \leq a_n \text{ for all } n \in \mathbb{N}, \quad (9)$$

$$\|v_n - p\| \leq b_n \text{ for all } n \in \mathbb{N}, \quad (10)$$

are available where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p .

Definition 3 [4] Let $(B, \|\cdot\|)$ be a Banach space. A map $T : B \rightarrow B$ is called weak-contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|y - Ty\|, \text{ for all } x, y \in B. \quad (11)$$

Definition 4 [4] Let $T, \tilde{T} : B \rightarrow B$ be two operators. We say that \tilde{T} is an approximate operator of T if for all $x \in B$ and for a fixed $\varepsilon > 0$ we have

$$\|Tx - \tilde{T}x\| \leq \varepsilon. \quad (12)$$

Lemma 1 [43] Let $\{\beta_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:

$$\beta_{n+1} \leq (1 - \lambda_n) \beta_n + \rho_n, \quad (13)$$

where $\lambda_n \in (0, 1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\frac{\rho_n}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} \beta_n = 0$.

Lemma 2 [39] Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the inequality

$$\beta_{n+1} \leq (1 - \mu_n) \beta_n + \mu_n \gamma_n, \quad (14)$$

where $\mu_n \in (0, 1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. Then the following inequality holds

$$0 \leq \limsup_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \gamma_n. \quad (15)$$

2 Main Results

Theorem 1 Let $T : D \rightarrow D$ be a weak-contraction map satisfying condition (11) with $F_T \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ an iterative sequence defined by (7) with real sequences $\{a_n^i\}_{n=0}^{\infty}$, $i \in \{1, 2\}$ in $[0, 1]$ satisfying $\sum_{k=0}^{\infty} a_k^1 a_k^2 = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a unique fixed point u^* of T .

Proof Uniqueness of u^* comes from condition (11). Using Picard-S iterative scheme (7) and condition (11), we obtain

$$\begin{aligned} \|z_n - u^*\| &\leq (1 - a_n^2) \|x_n - u^*\| + a_n^2 \|Tx_n - Tu^*\| \\ &\leq (1 - a_n^2) \|x_n - u^*\| + a_n^2 \delta \|x_n - u^*\| + a_n^2 L \|u^* - Tu^*\| \\ &= [1 - a_n^2 (1 - \delta)] \|x_n - u^*\|, \end{aligned} \quad (16)$$

$$\begin{aligned} \|y_n - u^*\| &\leq (1 - a_n^1) \|Tx_n - Tu^*\| + a_n^1 \|Tz_n - Tu^*\| \\ &\leq (1 - a_n^1) \delta \|x_n - u^*\| + a_n^1 \delta \|z_n - u^*\|, \end{aligned} \quad (17)$$

$$\|x_{n+1} - u^*\| \leq \delta \|y_n - u^*\|. \quad (18)$$

Combining (16), (17) and (18)

$$\|x_{n+1} - u^*\| \leq \delta^2 [1 - a_n^1 a_n^2 (1 - \delta)] \|x_n - u^*\|. \quad (19)$$

By induction

$$\begin{aligned} \|x_{n+1} - u^*\| &\leq \delta^{2(n+1)} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)] \|x_0 - u^*\| \\ &\leq \delta^{2(n+1)} \|x_0 - u^*\|^{n+1} e^{-(1-\delta) \sum_{k=0}^n a_k^1 a_k^2}. \end{aligned} \quad (20)$$

Since $\sum_{k=0}^{\infty} a_k^1 a_k^2 = \infty$,

$$e^{-(1-\delta) \sum_{k=0}^n a_k^1 a_k^2} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (21)$$

which implies $\lim_{n \rightarrow \infty} \|x_n - u^*\| = 0$.

Theorem 2 Let $T : D \rightarrow D$ with fixed point $u^* \in F_T \neq \emptyset$ be as in Theorem 1 and $\{q_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$ two iterative sequences defined by SP (6) and Picard-S (7) iteration methods with real sequences $\{a_n^i\}_{n=0}^{\infty}$, $i \in \{0, 1, 2\}$ in $[0, 1]$ satisfying $\sum_{k=0}^n a_k^1 a_k^2 = \infty$. Then the following are equivalent:

- (i) $\lim_{n \rightarrow \infty} \|x_n - u^*\| = 0$;
- (ii) $\lim_{n \rightarrow \infty} \|q_n - u^*\| = 0$.

Proof (i) \Rightarrow (ii): It follows from (6), (7), and condition (11) that

$$\begin{aligned} \|x_{n+1} - q_{n+1}\| &= \|(1 - a_n^0)(Ty_n - r_n) + a_n^0(Ty_n - Tr_n)\| \\ &\leq (1 - a_n^0) \|Ty_n - r_n\| + a_n^0 \|Ty_n - Tr_n\| \\ &\leq [1 - a_n^0 (1 - \delta)] \|y_n - r_n\| + [1 - a_n^0 (1 - L)] \|y_n - Ty_n\|, \end{aligned} \quad (22)$$

$$\begin{aligned} \|y_n - r_n\| &= \|(1 - a_n^1)(Tx_n - s_n) + a_n^1(Tz_n - Ts_n)\| \\ &\leq (1 - a_n^1) \|Tx_n - s_n\| + a_n^1 \|Tz_n - Ts_n\| \\ &\leq (1 - a_n^1) \|Tx_n - s_n\| + a_n^1 \delta \|z_n - s_n\| + a_n^1 L \|z_n - Tz_n\|, \end{aligned} \quad (23)$$

$$\begin{aligned} \|Tx_n - s_n\| &= \|(1 - a_n^2)(Tx_n - q_n) + a_n^2(Tx_n - Tq_n)\| \\ &\leq [1 - a_n^2 (1 - \delta)] \|x_n - q_n\| + [1 - a_n^2 (1 - L)] \|x_n - Tx_n\|, \end{aligned} \quad (24)$$

$$\begin{aligned} \|z_n - s_n\| &\leq (1 - a_n^2) \|x_n - q_n\| + a_n^2 \|Tx_n - Tq_n\| \\ &\leq [1 - a_n^2 (1 - \delta)] \|x_n - q_n\| + a_n^2 L \|x_n - Tx_n\|. \end{aligned} \quad (25)$$

Combining (22), (23), (24), and (25)

$$\begin{aligned} \|x_{n+1} - q_{n+1}\| &\leq [1 - a_n^0 (1 - \delta)] [1 - a_n^1 (1 - \delta)] [1 - a_n^2 (1 - \delta)] \|x_n - q_n\| \\ &\quad + [1 - a_n^0 (1 - \delta)] \{ (1 - a_n^1) [1 - a_n^2 (1 - L)] + a_n^1 a_n^2 \delta L \} \|x_n - Tx_n\| \\ &\quad + [1 - a_n^0 (1 - \delta)] a_n^1 L \|z_n - Tz_n\| + [1 - a_n^0 (1 - L)] \|y_n - Ty_n\|. \end{aligned} \quad (26)$$

It follows from the facts $\delta \in (0, 1)$ and $a_n^i \in [0, 1]$, $\forall n \in \mathbb{N}$, $i \in \{0, 1, 2\}$ that

$$[1 - a_n^0(1 - \delta)] [1 - a_n^1(1 - \delta)] [1 - a_n^2(1 - \delta)] < 1 - a_n^1 a_n^2 (1 - \delta). \quad (27)$$

Hence, inequality (26) becomes

$$\begin{aligned} \|x_{n+1} - q_{n+1}\| &\leq [1 - a_n^1 a_n^2 (1 - \delta)] \|x_n - q_n\| \\ &\quad + [1 - a_n^0(1 - \delta)] \{ (1 - a_n^1) [1 - a_n^2(1 - L)] + a_n^1 a_n^2 \delta L \} \|x_n - Tx_n\| \\ &\quad + [1 - a_n^0(1 - \delta)] a_n^1 L \|z_n - Tz_n\| + [1 - a_n^0(1 - L)] \|y_n - Ty_n\|. \end{aligned} \quad (28)$$

Denote that

$$\begin{aligned} \beta_n &:= \|x_n - q_n\|, \\ \lambda_n &:= a_n^1 a_n^2 (1 - \delta) \in (0, 1), \\ \rho_n &:= [1 - a_n^0(1 - \delta)] \{ (1 - a_n^1) [1 - a_n^2(1 - L)] + a_n^1 a_n^2 \delta L \} \|x_n - Tx_n\| \\ &\quad + [1 - a_n^0(1 - \delta)] a_n^1 L \|z_n - Tz_n\| + [1 - a_n^0(1 - L)] \|y_n - Ty_n\|. \end{aligned} \quad (29)$$

Since $\lim_{n \rightarrow \infty} \|x_n - u^*\| = 0$ and $Tu^* = u^*$

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = \lim_{n \rightarrow \infty} \|y_n - Ty_n\| = \lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0, \quad (30)$$

which implies $\frac{\rho_n}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, inequality (28) perform all assumptions in Lemma 1 and thus we obtain $\lim_{n \rightarrow \infty} \|x_n - q_n\| = 0$. Since

$$\|q_n - u^*\| \leq \|x_n - q_n\| + \|x_n - u^*\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (31)$$

$\lim_{n \rightarrow \infty} \|q_n - u^*\| = 0$.

(ii) \Rightarrow (i): It follows from (6), (7), and condition (11) that

$$\begin{aligned} \|q_{n+1} - x_{n+1}\| &= \|r_n - Ty_n + a_n^0(Tr_n - r_n)\| \\ &\leq \delta \|r_n - y_n\| + (1 + a_n^0 + L) \|r_n - Tr_n\|, \end{aligned} \quad (32)$$

$$\begin{aligned} \|r_n - y_n\| &\leq (1 - a_n^1) \|s_n - Tx_n\| + a_n^1 \|Ts_n - Tz_n\| \\ &\leq (1 - a_n^1) \|s_n - Tx_n\| + a_n^1 \delta \|s_n - z_n\| + a_n^1 L \|s_n - Ts_n\|, \end{aligned} \quad (33)$$

$$\begin{aligned} \|s_n - Tx_n\| &\leq \|Ts_n - Tx_n\| + \|s_n - Ts_n\| \\ &\leq \delta \|s_n - x_n\| + (1 + L) \|s_n - Ts_n\| \\ &\leq \delta \|q_n - x_n\| + \delta a_n^2 \|Tq_n - q_n\| + (1 + L) \|s_n - Ts_n\|, \end{aligned} \quad (34)$$

$$\begin{aligned} \|s_n - z_n\| &\leq (1 - a_n^2) \|q_n - x_n\| + a_n^2 \|Tq_n - Tx_n\| \\ &\leq [1 - a_n^2(1 - \delta)] \|q_n - x_n\| + a_n^2 L \|q_n - Tq_n\|. \end{aligned} \quad (35)$$

Combining (32), (33), (34), and (35)

$$\begin{aligned} \|q_{n+1} - x_{n+1}\| &\leq \delta^2 [1 - a_n^1 a_n^2 (1 - \delta)] \|q_n - x_n\| \\ &\quad + \delta^2 a_n^2 [1 - a_n^1 (1 - L)] \|q_n - Tq_n\| \\ &\quad + (1 + a_n^0 + L) \|r_n - Tr_n\| + \delta (1 - a_n^1 + L) \|s_n - Ts_n\|. \end{aligned} \quad (36)$$

Since $\delta \in (0, 1)$

$$\delta^2 [1 - a_n^1 a_n^2 (1 - \delta)] < 1 - a_n^1 a_n^2 (1 - \delta). \quad (37)$$

Hence, inequality (36) becomes

$$\begin{aligned} \|q_{n+1} - x_{n+1}\| &\leq [1 - a_n^1 a_n^2 (1 - \delta)] \|q_n - x_n\| \\ &\quad + \delta^2 a_n^2 [1 - a_n^1 (1 - L)] \|q_n - Tq_n\| \\ &\quad + (1 + a_n^0 + L) \|r_n - Tr_n\| + \delta (1 - a_n^1 + L) \|s_n - Ts_n\|. \end{aligned} \quad (38)$$

Denote that

$$\begin{aligned} \beta_n &:= \|q_n - x_n\|, \\ \lambda_n &:= a_n^1 a_n^2 (1 - \delta) \in (0, 1), \\ \rho_n &:= \delta^2 a_n^2 [1 - a_n^1 (1 - L)] \|q_n - Tq_n\| \\ &\quad + (1 + a_n^0 + L) \|r_n - Tr_n\| + \delta (1 - a_n^1 + L) \|s_n - Ts_n\|. \end{aligned} \quad (39)$$

Since $\lim_{n \rightarrow \infty} \|q_n - u^*\| = 0$ and $Tu^* = u^*$

$$\lim_{n \rightarrow \infty} \|q_n - Tq_n\| = \lim_{n \rightarrow \infty} \|r_n - Tr_n\| = \lim_{n \rightarrow \infty} \|s_n - Ts_n\| = 0, \quad (40)$$

which implies $\frac{\rho_n}{\lambda_n} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, inequality (38) perform all assumptions in Lemma 1 and thus we obtain $\lim_{n \rightarrow \infty} \|q_n - x_n\| = 0$. Since

$$\|x_n - u^*\| \leq \|q_n - x_n\| + \|q_n - u^*\| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (41)$$

$\lim_{n \rightarrow \infty} \|x_n - u^*\| = 0$.

Taking R. Chugh et al.'s result ([9], Corollary 3.2) into account, Theorem 2 leads to the following corollary under weaker assumption:

Corollary 1 Let $T : D \rightarrow D$ with fixed point $u^* \in F_T \neq \emptyset$ be as in Theorem

1. Then the followings are equivalent:

- 1) The Picard iteration method (3) converges to u^* ,
- 2) The Mann iteration method [25] converges to u^* ,
- 3) The Ishikawa iteration method [17] converges to u^* ,
- 4) The Noor iteration method (5) converges to u^* ,
- 5) S-iteration method [2] converges to u^* ,
- 6) The SP-iteration method (6) converges to u^* ,
- 7) CR-iteration method [9] converges to u^* ,
- 8) The Picard-S iteration method (7) converges to u^* .

Theorem 3 Let $T : D \rightarrow D$ with fixed point $u^* \in F_T \neq \emptyset$ be as in Theorem

1. Suppose that $\{\omega_n\}_{n=0}^\infty$, $\{q_n\}_{n=0}^\infty$ and $\{x_n\}_{n=0}^\infty$ are iterative sequences, respectively, defined by Noor (5), SP (6) and Picard-S (7) iterative schemes with real sequences $\{a_n^i\}_{n=0}^\infty \subset [0, 1]$, $i \in \{0, 1, 2\}$ satisfying

- (i) $0 \leq a_n^i < \frac{1}{1+\delta}$,
- (ii) $\lim_{n \rightarrow \infty} a_n^i = 0$.

Then the iterative sequence defined by (7) converges faster than the iterative sequences defined by (5) and (6) to a unique fixed point of T , provided that the initial point is the same for all iterations.

Proof From inequality (20), we have

$$\|x_{n+1} - u^*\| \leq \delta^{2(n+1)} \|x_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)] \quad (42)$$

Using (6) we obtain

$$\begin{aligned} \|q_{n+1} - u^*\| &= \|(1 - a_n^0) r_n + a_n^0 T r_n - u^*\| \\ &\geq (1 - a_n^0) \|r_n - u^*\| - a_n^0 \|T r_n - T u^*\| \\ &\geq [1 - a_n^0 (1 + \delta)] \|r_n - u^*\| \\ &\geq [1 - a_n^0 (1 + \delta)] \{ (1 - a_n^1) \|s_n - u^*\| - a_n^1 \delta \|s_n - u^*\| \} \\ &= [1 - a_n^0 (1 + \delta)] [1 - a_n^1 (1 + \delta)] \|s_n - u^*\| \\ &\geq [1 - a_n^0 (1 + \delta)] [1 - a_n^1 (1 + \delta)] \{ (1 - a_n^2) \|q_n - u^*\| - a_n^2 \delta \|q_n - u^*\| \} \\ &= [1 - a_n^0 (1 + \delta)] [1 - a_n^1 (1 + \delta)] [1 - a_n^2 (1 + \delta)] \|q_n - u^*\| \\ &\geq \dots \\ &\geq \|q_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^0 (1 + \delta)] [1 - a_k^1 (1 + \delta)] [1 - a_k^2 (1 + \delta)]. \end{aligned} \quad (43)$$

Using now (42) and (43)

$$\frac{\|x_{n+1} - u^*\|}{\|q_{n+1} - u^*\|} \leq \frac{\delta^{2(n+1)} \|x_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)]}{\|q_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^0 (1 + \delta)] [1 - a_k^1 (1 + \delta)] [1 - a_k^2 (1 + \delta)]}. \quad (44)$$

Define

$$\theta_n = \frac{\delta^{2(n+1)} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)]}{\prod_{k=0}^n [1 - a_k^0 (1 + \delta)] [1 - a_k^1 (1 + \delta)] [1 - a_k^2 (1 + \delta)]}. \quad (45)$$

By the assumption

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\theta_{n+1}}{\theta_n} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{\delta^{2(n+2)} \prod_{k=0}^{n+1} [1 - a_k^1 a_k^2 (1 - \delta)]}{\prod_{k=0}^{n+1} [1 - a_k^0 (1 + \delta)] [1 - a_k^1 (1 + \delta)] [1 - a_k^2 (1 + \delta)]}}{\frac{\delta^{2(n+1)} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)]}{\prod_{k=0}^n [1 - a_k^0 (1 + \delta)] [1 - a_k^1 (1 + \delta)] [1 - a_k^2 (1 + \delta)]}} \\ &= \lim_{n \rightarrow \infty} \frac{\delta^2 [1 - a_{n+1}^1 a_{n+1}^2 (1 - \delta)]}{[1 - a_{n+1}^0 (1 + \delta)] [1 - a_{n+1}^1 (1 + \delta)] [1 - a_{n+1}^2 (1 + \delta)]} \\ &= \delta^2 < 1. \end{aligned} \quad (46)$$

It thus follows from ratio test that $\sum_{n=0}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n \rightarrow \infty} \theta_n = 0$ which implies that the iterative sequence defined by (7) converges faster than the iterative sequence defined by SP iteration method (6).

Using Noor iteration method (5), we get

$$\begin{aligned}
\|\omega_{n+1} - u^*\| &= \|(1 - a_n^0)\omega_n + a_n^0 T\varpi_n - u^*\| \\
&\geq (1 - a_n^0)\|\omega_n - u^*\| - a_n^0 \|T\varpi_n - Tu^*\| \\
&\geq (1 - a_n^0)\|\omega_n - u^*\| - a_n^0 \delta \|\varpi_n - u^*\| \\
&\geq [1 - a_n^0 - a_n^0 \delta (1 - a_n^1)]\|\omega_n - u^*\| - a_n^0 a_n^1 \delta^2 \|\rho_n - u^*\| \\
&\geq \{1 - a_n^0 - a_n^0 \delta (1 - a_n^1) - a_n^0 a_n^1 \delta^2 [1 - a_n^2 (1 - \delta)]\}\|\omega_n - u^*\| \\
&\geq \{1 - a_n^0 - a_n^0 \delta [1 - a_n^1 (1 - \delta)]\}\|\omega_n - u^*\| \\
&\geq [1 - a_n^0 (1 + \delta)]\|\omega_n - u^*\| \\
&\geq \dots \\
&\geq \|\omega_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^0 (1 + \delta)].
\end{aligned} \tag{47}$$

It follows by (42) and (47) that

$$\frac{\|x_{n+1} - u^*\|}{\|\omega_{n+1} - x^*\|} \leq \frac{\delta^{2(n+1)} \|x_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)]}{\|\omega_0 - u^*\|^{n+1} \prod_{k=0}^n [1 - a_k^0 (1 + \delta)]}. \tag{48}$$

Define

$$\theta_n = \frac{\delta^{2(n+1)} \prod_{k=0}^n [1 - a_k^1 a_k^2 (1 - \delta)]}{\prod_{k=0}^n [1 - a_k^0 (1 + \delta)]}. \tag{49}$$

By the assumption

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{\theta_{n+1}}{\theta_n} &= \lim_{n \rightarrow \infty} \frac{\delta^2 [1 - a_{n+1}^1 a_{n+1}^2 (1 - \delta)]}{[1 - a_{n+1}^0 (1 + \delta)]} \\
&= \delta^2 < 1.
\end{aligned} \tag{50}$$

It thus follows from ratio test that $\sum_{n=0}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n \rightarrow \infty} \theta_n = 0$ which implies that the iterative sequence defined by (7) converges faster than the iterative sequence defined by Noor iteration method (5).

By use of the following example due to [45], it was shown in ([9], Example 4.1) that CR iterative method [9] is faster than all of Picard (3), S [2], Noor (5) and SP (6) iterative methods for a particular class of operators which is included in the class of weak-contraction mappings satisfying (11). In the following, for the sake of consistent comparison, we will use the same example as that of ([9], Example 4.1) in order to compare the rates

of convergence between the Picard-S iterative scheme (7) and the CR iteration method [9] for weak-contraction mappings. In the following example, for convenience, we use the notations (PS_n) and (CR_n) for the iterative sequences associated to Picard-S (7) and CR [9] iteration methods, respectively.

Example 1 [45] Define a mapping $T : [0, 1] \rightarrow [0, 1]$ as $Tx = \frac{x}{2}$. Let $a_n^0 = a_n^1 = a_n^2 = 0$, for $n = 1, 2, \dots, 24$ and $a_n^0 = a_n^1 = a_n^2 = \frac{4}{\sqrt{n}}$, for all $n \geq 25$.

It can be seen easily that the mapping T satisfies condition (11) with the unique fixed point $0 \in F_T$. Furthermore, it is easy to see that Example 1 satisfies all the conditions of Theorem 1. Indeed, let $x_0 \neq 0$ be an initial point for the iterative sequences (PS_n) and (CR_n) . Utilizing Picard-S (7) and CR [9] iteration methods we obtain

$$\begin{aligned} PS_n &= \frac{1}{2} \left(\frac{1}{2} - \frac{4}{n} \right) x_n \\ &= \dots \\ &= \prod_{k=25}^n \left(\frac{1}{4} - \frac{2}{k} \right) x_0, \end{aligned} \quad (51)$$

$$\begin{aligned} CR_n &= \left(\frac{1}{2} - \frac{1}{\sqrt{n}} - \frac{4}{n} + \frac{8}{n\sqrt{n}} \right) x_n \\ &= \dots \\ &= \prod_{k=25}^n \left(\frac{1}{2} - \frac{1}{\sqrt{k}} - \frac{4}{k} + \frac{8}{k\sqrt{k}} \right) x_0. \end{aligned} \quad (52)$$

It follows from (51) and (52) that

$$\begin{aligned} \frac{|PS_n - 0|}{|CR_n - 0|} &= \frac{\prod_{k=25}^n \left(\frac{1}{4} - \frac{2}{k} \right) x_0}{\prod_{k=25}^n \left(\frac{1}{2} - \frac{1}{\sqrt{k}} - \frac{4}{k} + \frac{8}{k\sqrt{k}} \right) x_0} \\ &= \prod_{k=25}^n \frac{\frac{1}{4} - \frac{2}{k}}{\frac{1}{2} - \frac{1}{\sqrt{k}} - \frac{4}{k} + \frac{8}{k\sqrt{k}}} \\ &= \prod_{k=25}^n \frac{(k-8)\sqrt{k}}{2(k\sqrt{k} - 2k - 8\sqrt{k} + 16)} \\ &= \prod_{k=25}^n \frac{(k-8)\sqrt{k}}{2(\sqrt{k}-2)(k-8)} \\ &= \prod_{k=25}^n \frac{\sqrt{k}}{2(\sqrt{k}-2)}. \end{aligned} \quad (53)$$

For all $k \geq 25$, we have

$$\begin{aligned}
\frac{(k-2)(\sqrt{k}-4)}{4} &> 1 \\
&\Rightarrow (k-2)(\sqrt{k}-4) > 4 \\
&\Rightarrow k(\sqrt{k}-4) > 2(\sqrt{k}-2) \\
&\Rightarrow \frac{\sqrt{k}-4}{2(\sqrt{k}-2)} > \frac{1}{k} \\
&\Rightarrow \frac{\sqrt{k}}{2(\sqrt{k}-2)} < 1 - \frac{1}{k}, \tag{54}
\end{aligned}$$

which yields

$$\frac{|PS_n - 0|}{|CR_n - 0|} = \prod_{k=25}^n \frac{\sqrt{k}}{2(\sqrt{k}-2)} < \prod_{k=25}^n \left(1 - \frac{1}{k}\right) = \frac{24}{n}. \tag{55}$$

Therefore, we have

$$\lim_{n \rightarrow \infty} \frac{|PS_n - 0|}{|CR_n - 0|} = 0, \tag{56}$$

which implies that the Picard-S iterative scheme (7) is faster than the CR iteration method [9].

Having regard to R. Chugh et al.'s result ([9], Example 4.1), L.B. Ćirić et al.'s results [10] and Example 1 above, we conclude that Picard-S iteration method is faster than all Picard (3), Mann [25], Ishikawa [17], S [2], Noor (5) and SP (6) iterative methods.

We are now able to establish the following data dependence result.

Theorem 4 Let T with fixed point $u^* \in F_T \neq \emptyset$ be as in Theorem 1 and \tilde{T} an approximate operator of T . Let $\{x_n\}_{n=0}^\infty$ be an iterative sequence generated by (7) for T and define an iterative sequence $\{\tilde{x}_n\}_{n=0}^\infty$ as follows

$$\begin{cases} \tilde{x}_0 \in D, \\ \tilde{x}_{n+1} = \tilde{T}\tilde{y}_n, \\ \tilde{y}_n = (1 - a_n^1)\tilde{T}\tilde{x}_n + a_n^1\tilde{T}\tilde{z}_n, \\ \tilde{z}_n = (1 - a_n^2)\tilde{x}_n + a_n^2\tilde{T}\tilde{x}_n, \quad n \in \mathbb{N}, \end{cases} \tag{57}$$

where $\{a_n^i\}_{n=0}^\infty$, $i \in \{1, 2\}$ be real sequences in $[0, 1]$ satisfying (i) $\frac{1}{2} \leq a_n^1 a_n^2$ for all $n \in \mathbb{N}$, and (ii) $\sum_{n=0}^\infty a_n^1 a_n^2 = \infty$. If $\tilde{T}\tilde{u}^* = \tilde{u}^*$ such that $\tilde{x}_n \rightarrow \tilde{u}^*$ as $n \rightarrow \infty$, then we have

$$\|u^* - \tilde{u}^*\| \leq \frac{5\varepsilon}{1 - \delta}, \tag{58}$$

where $\varepsilon > 0$ is a fixed number.

Proof It follows from (7), (11), (12), and (57) that

$$\begin{aligned} \|z_n - \tilde{z}_n\| &\leq (1 - a_n^2) \|x_n - \tilde{x}_n\| + a_n^2 \|Tx_n - \tilde{T}\tilde{x}_n\| \\ &\leq [1 - a_n^2 + a_n^2 \delta] \|x_n - \tilde{x}_n\| + a_n^2 L \|x_n - Tx_n\| + a_n^2 \varepsilon, \end{aligned} \quad (59)$$

$$\begin{aligned} \|y_n - \tilde{y}_n\| &\leq (1 - a_n^1) \delta \|x_n - \tilde{x}_n\| + a_n^1 \delta \|z_n - \tilde{z}_n\| \\ &\quad + (1 - a_n^1) L \|x_n - Tx_n\| + a_n^1 L \|z_n - Tz_n\| \\ &\quad + (1 - a_n^1) \varepsilon + a_n^1 \varepsilon, \end{aligned} \quad (60)$$

$$\|x_{n+1} - \tilde{x}_{n+1}\| \leq \delta \|y_n - \tilde{y}_n\| + L \|y_n - Ty_n\| + \varepsilon. \quad (61)$$

From the relations (59), (60), and (61)

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &\leq \delta^2 [1 - a_n^1 a_n^2 (1 - \delta)] \|x_n - \tilde{x}_n\| \\ &\quad + \{a_n^1 a_n^2 \delta^2 L + (1 - a_n^1) \delta L\} \|x_n - Tx_n\| \\ &\quad + L \|y_n - Ty_n\| + a_n^1 \delta L \|z_n - Tz_n\| \\ &\quad + a_n^1 a_n^2 \delta^2 \varepsilon + (1 - a_n^1) \delta \varepsilon + a_n^1 \delta \varepsilon + \varepsilon. \end{aligned} \quad (62)$$

Since $a_n^1, a_n^2 \in [0, 1]$ and $\frac{1}{2} \leq a_n^1 a_n^2$ for all $n \in \mathbb{N}$

$$1 - a_n^1 a_n^2 \leq a_n^1 a_n^2, \quad (63)$$

$$1 - a_n^1 \leq 1 - a_n^1 a_n^2 \leq a_n^1 a_n^2, \quad (64)$$

$$1 \leq 2a_n^1 a_n^2. \quad (65)$$

Use of the facts $\delta, \delta^2 \in (0, 1)$, (63), (64), and (65) in (62) yields

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &\leq [1 - a_n^1 a_n^2 (1 - \delta)] \|x_n - \tilde{x}_n\| \\ &\quad + a_n^1 a_n^2 (1 - \delta) \left\{ \frac{L\delta(1 + \delta) \|x_n - Tx_n\|}{1 - \delta} \right. \\ &\quad \left. + \frac{2L \|y_n - Ty_n\| + 2\delta L \|z_n - Tz_n\| + 5\varepsilon}{1 - \delta} \right\}. \end{aligned} \quad (66)$$

Define

$$\beta_n := \|x_n - \tilde{x}_n\|, \quad (67)$$

$$\mu_n := a_n^1 a_n^2 (1 - \delta) \in (0, 1),$$

$$\gamma_n := \frac{L\delta(1 + \delta) \|x_n - Tx_n\| + 2L \|y_n - Ty_n\| + 2\delta L \|z_n - Tz_n\| + 5\varepsilon}{1 - \delta} \geq 0.$$

Hence, the inequality (66) perform all assumptions in Lemma 2 and thus an application of Lemma 2 to (66) yields

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \|x_n - \tilde{x}_n\| \\ &\leq \limsup_{n \rightarrow \infty} \frac{L\delta(1 + \delta) \|x_n - Tx_n\| + 2L \|y_n - Ty_n\| + 2\delta L \|z_n - Tz_n\| + 5\varepsilon}{1 - \delta}. \end{aligned} \quad (68)$$

We know from Theorem 1 that $\lim_{n \rightarrow \infty} x_n = u^*$ and since $Tu^* = u^*$

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = \lim_{n \rightarrow \infty} \|y_n - Ty_n\| = \lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0. \quad (69)$$

Therefore the inequality (68) becomes

$$\|u^* - \tilde{u}^*\| \leq \frac{5\varepsilon}{1 - \delta}. \quad (70)$$

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